JHEP HYPER VERSION

RECEIVED: *January 31, 2000*, ACCEPTED: *March 1, 2000*

Tension is dimension

Jeffrey A. Harvey

Enrico Fermi Institute and Department of Physics, University of Chicago
5640 Ellis Avenue, Chicago, IL 60637, USA
E-mail: `harvey@theory.uchicago.edu`

Shamit Kachru

Department of Physics and SLAC, Stanford University
Stanford, CA 94305/94309, USA
E-mail: `skachru@leland.stanford.edu`

Gregory Moore*

Department of Physics, Yale University
Box 208120, New Haven, CT 06520, USA
E-mail: `moore@castalia.physics.yale.edu`

Eva Silverstein

Department of Physics and SLAC, Stanford University
Stanford, CA 94305/94309, USA
E-mail: `evas@SLAC.Stanford.EDU`

ABSTRACT: We propose a simple universal formula for the tension of a D-brane in terms of a regularized dimension of the associated conformal field theory statespace.

KEYWORDS: D-branes, Conformal Field Models in String Theory.

*Address after January 1, 2000: Dept. of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855-0849, USA

JHEP03(2000)001

Contents

1. Introduction

D-branes are destined to play a fundamental role in the formulation of nonperturbative string theory. Nevertheless, despite much work, and a good understanding of examples such as toroidal compactification, there is as yet no general formulation of the D-brane spectrum applicable to an arbitrary closed string background. In this note we point out one simple aspect of D-branes which, we conjecture, is quite general. Namely, the square of the tension of a D-brane is proportional to the regularized dimension of some infinite dimensional algebra. We show that this is indeed true for the wide class of D-branes associated to rational and quasi-rational conformal field theories.

While the general construction of the D-brane spectrum has yet to be carried out, much is indeed known thanks to recent vigorous development of boundary conformal field theory, and the boundary state formalism. A partial list of references includes [1]–[14]. One point which is well-established is that boundary states $|B\rangle\rangle$ are linear functionals on the closed string statespace $\mathcal{H}^{\text{closed}}$ preserving conformal invariance

$$(L_n - \tilde{L}_{-n})|B\rangle\rangle = 0\,, \tag{1.1}$$

where $L_n, \tilde{L}_n$ are the left and right Virasoro generators.

One might think that (1.1), which imposes conformal invariance on the string worldsheet, is the only condition one needs to impose to find physically acceptable boundary states in string theory. Taking this point of view immediately leads to problems: there are far too many solutions to (1.1) for an acceptable D-brane spectrum. Indeed, (1.1) is a linear equation and solutions are in one to one correspondence with spinless Virasoro primaries $\phi^{\alpha}_{h,\tilde{h}}$ in the closed string spectrum [15] (here $h=\tilde{h}$ and α is a degeneracy index.)

This result is most naturally understood as follows [5, 14]. We can decompose $\mathcal{H}^{\rm closed}$ in terms of Virasoro irreps V_h as

$$\mathcal{H}^{\rm closed} = \oplus_{(h,\tilde{h})\in {\rm Spec}} V_h \otimes \tilde{V}_{\tilde{h}}\,, \tag{1.2}$$

where the spectrum of the closed string theory is determined by the set Spec of pairs of conformal dimensions. We can then solve (1.1) in each component of $\mathcal{H}^{\rm closed}$ as

$$|B\rangle\rangle_{h,\tilde{h}} = \sum_{m,\tilde{m}} a_{m,\tilde{m}} |h,m\rangle \otimes |\tilde{h},\tilde{m}\rangle \in V_h \otimes \tilde{V}_{\tilde{h}}\,. \tag{1.3}$$

Using the inner product on $\tilde{V}_{\tilde{h}}$ under which $L_n^\dagger = L_{-n}$, $|B\rangle\rangle_{h,\tilde{h}}$ is equivalent to a homomorphism $B_{h,\tilde{h}}$: $\tilde{V}_{\tilde{h}} \to V_h$ given by

$$B_{h,\tilde{h}} = \sum_{m,\tilde{m}} a_{m,\tilde{m}} |h,m\rangle \otimes \langle \tilde{h},\tilde{m}| \tag{1.4}$$

obeying $L_n B_{h,\tilde{h}} = B_{h,\tilde{h}} \tilde{L}_n$. That is, $B_{h,\tilde{h}}$ is an intertwiner between V_h and $\tilde{V}_{\tilde{h}}$. Since these are irreps it follows from Schur's lemma that that $|B\rangle\rangle_{h,\tilde{h}}$ vanishes if $h \neq \tilde{h}$ and that $B_{h,\tilde{h}}$ is proportional to the identity when $h=\tilde{h}$ (the fact that we need $h=\tilde{h}$ is already clear from the $n=0$ component of (1.1)). Choosing the proportionality constant to be one, we can take B acting on $\mathcal{H}^{\rm closed}$ to be the projection operator onto $V_h \otimes \tilde{V}_{\tilde{h}=h}$. We denote the corresponding "states" in $\mathcal{H}^{\rm closed}$ by $|h,\tilde{h}=h\rangle\rangle$. There is a natural generalization of this construction to arbitrary chiral algebras [16, 12, 14] and in this context the states are often referred to as "Ishibashi states". We will refer to them as "character states".

One problem with these Virasoro character states is that most of them do not couple to the graviton at leading order in string perturbation theory. This follows because the overlap with the graviton state of a primary or descendent in another irrep (not containing the graviton state as primary) is zero. Therefore, if one wrapped these "branes" on cycles the resulting particles would have string scale $\mathcal{O}(g_s^0)$ masses and one would expect severe problems with unitarity.

On the other hand, it has been well appreciated for some time that in addition to (1.1) various sewing and locality conditions should be imposed [4, 6], at least if one desires a description in terms of local boundary conformal field theory on the string worldsheet. An especially important role is played by the Cardy condition [3]. To state this condition consider possible boundary states $|\alpha\rangle\rangle$, $|\beta\rangle\rangle$, and compute the partition function (cylinder amplitude)

$$Z_{\alpha\beta} = \langle\langle \beta | q_c^{\frac{1}{2}(L_0^c + \tilde{L}_0^c - \frac{c}{12})} |\alpha\rangle\rangle\,, \tag{1.5}$$

where $L_0^c, \tilde{L}_0^c$ are the left and right-moving closed string hamiltonians and $q_c = e^{-2\pi t_c}$. This can be given a hamiltonian interpretation in the open string channel by viewing the cylinder as an annulus with euclidean time running around the annulus. After a conformal rescaling of coordinates we should thus be able to write (1.5) as

$$\mathrm{Tr}_{H_{\alpha\beta}}\, q_o^{L_0^{\mathrm{open}}-c/24} \tag{1.6}$$

where $H_{\alpha\beta}$ is the Hilbert space of open strings with boundary conditions defined by α, β and $q_o = e^{-2\pi t_o} = e^{-2\pi/t_c}$. Cardy's condition follows from the equality of (1.5) and (1.6). In other words, the modular transform of (1.5) to the variable q_o should have a q_o expansion with non-negative integer coefficients for all possible pairs of boundary states $|\alpha\rangle\rangle, |\beta\rangle\rangle$. We will call such a set of boundary states a Cardy set.

In the following we demonstrate that imposing the Cardy condition reduces the solutions to (1.1) down to an acceptable few: boundary states satisfying the Cardy condition must couple to the graviton at leading order in string perturbation theory, and therefore have tensions of order $\mathcal{O}(g_s^{-1})$.

Moreover, we find that the tensions are given by the (suitably regularized) dimension of an associated open string statespace as in equations (2.3) and (4.1) below. We believe that the generality of this result has not been appreciated previously, although many of the elements of our argument are not new. In particular, the regularized dimension has appeared previously in the literature on boundary CFT as the boundary entropy of Affleck and Ludwig [17]. The connection between D-brane energy and boundary entropy was made in [18] for D-branes moving in flat space. Furthermore, we discovered after completing this work that our computation of the boundary entropy for torus compactifications in section 2.2 below appeared earlier in [19].

2. The general argument: bosonic string

We consider a spacetime defined by a closed conformal field theory of the type $\mathcal{C}(\mathbb{R}^{1,25-d}) \otimes \mathcal{C}_2$. The first factor is the usual conformal field theory of $26-d$ free uncompactified bosons and ghosts. For our purposes it suffices to work in light-cone gauge in which case we can drop the ghost fields and view the first factor as the CFT of $24-d$ free bosons. The second factor is an arbitrary unitary CFT of $c = \bar{c} = d$. We will choose d sufficiently large so that we can view the Dp-brane we are interested in as a D-particle in $26-d$ spacetime dimensions.

2.1 D-brane Tension

e consider D-brane boundary states of the form:

$$|x\rangle\rangle \otimes |\alpha\rangle\rangle\,. \tag{2.1}$$

Here $|\alpha\rangle\rangle$ is a boundary state for $\mathcal{C}_2$ which is assumed to be part of a Cardy set. It corresponds to an open string channel statespace $\mathcal{H}_{\alpha\alpha}$ as in (1.6). The first factor

$|x\rangle\rangle$ is a standard position eigenstate D-brane state constructed from coherent states of lightcone gauge oscillators.

$$|x\rangle\rangle := \mathcal{N}_{\rm s.t.} \int d^{24-d}k e^{ikx} e^{-\sum \frac{1}{n}\alpha_{-n}\tilde{\alpha}_{-n}} |k\rangle \,. \tag{2.2}$$

Here $|k\rangle$ denotes a momentum eigenstate in the closed string Hilbert space; the normalization $\mathcal{N}_{\rm s.t.}$ can be gleaned from [8] and involves powers of $2, \pi, \ell_s$.

The D-brane state (2.1) describes a particle in the $(26-d)$-dimensional spacetime theory. The formula for the mass of this particle in terms of $\mathcal{H}_{\alpha\alpha}$ is:

$$(\ell_s M)^2 = \frac{1}{(64\pi)^2} \frac{(2\pi\ell_s)^{24-d}}{G_{26-d}} \dim \mathcal{H}_{\alpha\alpha} \,, \tag{2.3}$$

where G_{26-d} is the Newton constant in 26-d dimensions, ℓ_s is the string length ($\alpha' = \ell_s^2$) and $\dim \mathcal{H}_{\alpha\alpha}$ is the regularized dimension, defined by

$$\dim \mathcal{H}_{\alpha\alpha} = \lim_{\tau\to 0} e^{2\pi i c/24(-1/\tau)} \, {\rm Tr}_{\mathcal{H}_{\alpha\alpha}} q^{L_0 - c/24} \,, \tag{2.4}$$

where $q = e^{2\pi i \tau}$.

In order to prove (2.3) we will assume that $\mathcal{C}_2$ is described by rational conformal field theory (RCFT). We believe that this is only a technical assumption and that (2.3) holds more generally. Some evidence for this will be given later where (2.3) will be seen to hold in quasirational theories.

In RCFT we have isomorphic left and right-moving chiral algebras $\mathcal{A}_L = \mathcal{A}_R = \mathcal{A}$ which contain the Virasoro algebra and which may in general be subalgebras of a larger chiral algebra. We denote the moments of the chiral fields generating $\mathcal{A}$ by W_n. By definition, the Hilbert space can be decomposed into a finite set of irreps V_j of $\mathcal{A}$

$$\mathcal{H}^{\rm closed} = \oplus_{(j,\tilde{j})\in {\rm Spec}} V_j \otimes \tilde{V}_{\tilde{j}} \,, \tag{2.5}$$

with Spec labelling the irreps in the spectrum, possibly with multiplicities $N_{j,\tilde{j}}$. The characters

$$\chi_i(\tau) = {\rm Tr}_{V_i} \, q^{L_0 - c/24} \tag{2.6}$$

transform under modular transformations $\tau \to -1/\tau$ according to

$$\chi_i\left(-\frac{1}{\tau}\right) = \sum_j S_i^{\ j} \chi_j(\tau) \,. \tag{2.7}$$

For each irrep V_i there is a primary field ϕ_i obeying the fusion algebra

$$\phi_i \times \phi_j = N_{ij}^{\ \ k} \phi_k \,, \tag{2.8}$$

with structure constants related to the S matrix by the Verlinde formula

$$N_{ijk} = \sum_\ell \frac{S_i^{\ \ell} S_j^{\ \ell} S_k^{\ \ell}}{S_1^{\ \ell}} \,. \tag{2.9}$$

Demanding that $\mathcal{A}$ act in the RCFT with boundary requires that

$$(W_n - (-1)^{h_W}\Omega(\bar{W}_{-n}))|\alpha\rangle\rangle = 0\,, \tag{2.10}$$

where h_W is the conformal dimension (spin) of W and Ω is an automorphism of $\mathcal{A}$. Character states $|j\rangle\rangle$ solving (2.10) can be constructed by a slight variant of the argument given earlier [16, 14]. The character states $|j\rangle\rangle$ of $\mathcal{A}$ do not in general form a Cardy set. Since character states form a basis, we can write possible elements of a Cardy set as

$$|\alpha\rangle\rangle = \sum_j \frac{\psi_\alpha^j}{\sqrt{S_1{}^j}}|j\rangle\rangle\,, \tag{2.11}$$

where the factor in the denominator has been put in for later convenience. It is positive, see below.

Using

$$\langle\langle j|q_c^{\frac{1}{2}(L_0+\tilde{L}_0-\frac{c}{12})}|k\rangle\rangle \;=\; \delta_{jk}\chi_j(q_c) \tag{2.12}$$

we then have

$$Z_{\beta\alpha} = \sum_j \frac{\psi_\alpha^j(\psi_\beta^j)^*}{S_1{}^j}\chi_j(q_c) = \sum_{j,k}\frac{\psi_\alpha^j(\psi_\beta^j)^*}{S_1{}^j}S_j{}^k\chi_k(q_o)\,. \tag{2.13}$$

On the other hand, since $\mathcal{A}$ acts on the open string Hilbert space, we can also decompose $\mathcal{H}_{\alpha\beta}$ into $\mathcal{A}$ irreps so that

$$\mathrm{Tr}_{H_{\alpha\beta}}\, q_o^{L_0-c/24} = \sum_i (n^i)_{\alpha\beta}\chi_i(q_o)\,, \tag{2.14}$$

with $n^i_{\alpha\beta}$ non-negative integers. Equating (2.14) and (2.13) then gives Cardy's condition in the context of boundary RCFT:

$$(n^i)_{\alpha\beta} = \sum_j \frac{S_i{}^j}{S_1{}^j}\psi_\alpha^j(\psi_\beta^j)^*\,. \tag{2.15}$$

The solution originally given by Cardy has α running over the irreps of $\mathcal{A}$ and $(n^i)_{\alpha\beta} = N^i_{\alpha\beta}$, $\psi_{\alpha j} = S_{\alpha j}$, which clearly solves (2.15) using (2.9). However, in general other solutions will exist [6, 11, 14] (this happens for as simple a system as the rational circle) and we will only assume (2.15) in what follows.

We are now ready to prove (2.3). The key observation is that the mass is measured by the one-point function with the graviton. The graviton vertex operator $\epsilon_{\mu\nu}\partial X^\mu\bar{\partial}X^\nu e^{ik\cdot X}\otimes 1$ is the unit operator in the internal theory $\mathcal{C}_2$. Thus we need only know the coupling of $|\alpha\rangle\rangle$ to the character of the representation with the unit operator in order to compute the mass of the D-brane.

The unit operator is in a unique character state $|1\rangle\rangle$ and uniqueness of the vacuum implies $\langle 0|1\rangle\rangle = 1$. Therefore, the dependence of the tension on the internal conformal field theory is exactly $\psi^1_\alpha/\sqrt{S_{11}}$. On the other hand, the regularized dimension of $\mathcal{H}_{\alpha\alpha}$ is

$$\begin{aligned}\dim \mathcal{H}_{\alpha\alpha} &= \lim_{q_o\to 1} q_c^{c/24}\, \mathrm{Tr}_{\mathcal{H}_{\alpha\alpha}}\, q_o^{L_0 - c/24} \\ &= \lim_{q_o\to 1} q_c^{c/24} \sum_{j,k} \frac{S_k{}^j}{S_1{}^k} |\psi^k_\alpha|^2 \chi_j(q_o) \\ &= \frac{|\psi^1_\alpha|^2}{S_{11}}\,,\end{aligned} \tag{2.16}$$

where we have used (2.14) and (2.15), and the fact that $\chi_k(q_c)$ goes like $q_c^{h_k - c/24}$ as $q_c \to 0$ so that the dominant contribution comes from the identity representation with $h_1 = 0$ in the limit. This then proves (2.3) up to overall factors which can be determined by working on the torus (see below) and comparing to [20].

It is easy to show that the dimension (and hence the tension) is nonzero, at least in RCFTs. In the Cardy condition we impose positivity $(n^j)_{\alpha\beta} \geq 0$ and the vacuum must appear in $(n^0)_{\alpha\alpha} \geq 1$. Moreover, the matrix element $S_{j1} \geq 0$. This latter fact is easily proved since the regularized dimension of the representation j of the chiral algebra is given by the modular matrix [21]:

$$\dim \mathcal{H}_j = \lim_{t\to 0} \frac{\mathrm{Tr}_{\mathcal{H}_j}\, e^{-2\pi t H}}{\mathrm{Tr}_{\mathcal{H}_1}\, e^{-2\pi t H}} = \frac{S_{j1}}{S_{11}}\,. \tag{2.17}$$

This is a limit of positive quantities and hence nonnegative.[1] In fact, in RCFT the S_{j1} cannot vanish because that leads to inconsistencies in the modular representation. Indeed the interpretation of this quantity as a positive dimension is crucial to the general picture of RCFT as a generalization of group theory [22, 23].

Remarks:

1. From the closed string point of view, the limit used to define the regularized dimension is of course the same as the limit originally used by Polchinski to compute the tension for D-branes in flat space [24].

2. The derivation is valid to leading order in string perturbation theory, which makes it an *exact* statement for BPS branes. It would be interesting to see if there is a sense in which it is true beyond leading order. Because the result seems very natural to us, we conjecture that it will continue to hold even for nonrational backgrounds.

3. Note that (2.3) behaves nicely upon inclusion of Chan-Paton spaces

$$\sqrt{\dim(\mathcal{H}_{\alpha\alpha} \otimes \mathrm{Mat}_N(\mathbb{R}))} = N\sqrt{\dim(\mathcal{H}_{\alpha\alpha})} \tag{2.18}$$

[1] We are assuming unitarity of the internal CFT $\mathcal{C}_2$ here, which is reasonable for string theory applications.

Actually, in string theory we impose a reality condition on Chan-Paton factors in $\mathrm{Mat}_N(\mathbb{C})$ so that Chan-Paton factors take values in the hermitean matrices, H_N, but the real dimension is again $\dim H_N = N^2$.

4. As mentioned earlier, the quantity $\psi_\alpha^{\ 1}/\sqrt{S_{11}}$ is what Affleck and Ludwig call the "nonintegral groundstate degeneracy" in boundary CFT. It has been conjectured, and established in conformal perturbation theory, that this quantity descreases along renormalization group flows. See [17, 25, 14] for further discussion.

2.2 Example: compactification on tori

A simple example of the above rule is provided by the gaussian model on T^d with constant background metric $G_{\mu\nu}$ and two-form $B_{\mu\nu}$. We set $E = G + B$. (If we pick rational values for the Narain moduli, this case fits simply into the framework discussed above; in the general quasirational case we will take some shortcuts below in showing that the squared tension is given by a regularized dimension, given boundary states satisfying the Cardy condition.) The closed conformal field theory is characterized by a Narain lattice $\Gamma(E) \subset \mathbb{R}^{d,d}$. The isomorphism of $u(1)^d$ left and right chiral algebras is given by $\alpha_n = \mathcal{R} \cdot \tilde{\alpha}_{-n}$, where $\mathcal{R} \in O(d;\mathbb{R})$ is some rotation matrix [1, 11, 12].

Let

$$\Lambda = \Gamma(E) \cap \{(p_L; p_R) : p_L = \mathcal{R} \cdot p_R\} \,. \tag{2.19}$$

We denote the rank of Λ by r and the metric tensor on Λ by $\mathcal{G}_{ij}$. By this we mean the metric appearing in the (euclidean) inner product $p_L^2 + p_R^2 = n^T \mathcal{G} n$ for a d dimensional vector of integers n. Note that Λ depends on both E and $\mathcal{R}$.

Let θ be a character of Λ. Cardy states will be of the form

$$|\theta; E, \mathcal{R}\rangle\rangle = N_\theta \sum_{\lambda \in \Lambda} e^{2\pi i \theta \cdot \lambda} e^{-S(\mathcal{R})} |\lambda\rangle \tag{2.20}$$

where $S(\mathcal{R}) = \sum_n \alpha_{-n} \mathcal{R} \tilde{\alpha}_{-n}/n$ and we use the discrete measure for the momentum eigenvectors. Let us ask that the single state $|\theta; E, \mathcal{R}\rangle\rangle$ form a Cardy state. We compute

$$\langle\langle\theta| q_c^{\frac{1}{2}(L_0 + \tilde{L}_0 - c/12)} |\theta\rangle\rangle = \frac{|N_\theta|^2}{\eta(q_c)^d} \sum_{\lambda \in \Lambda} q_c^{\frac{1}{2}\lambda_L^2} = \frac{|N_\theta|^2}{\eta(q_c)^d} \sum_{n \in \mathbb{Z}^r} e^{-\pi t_c n^i \mathcal{G}_{ij} n^j} \,. \tag{2.21}$$

From Poisson resummation we get

$$\langle\langle\theta| q_c^{\frac{1}{2}(L_0 + \tilde{L}_0 - c/12)} |\theta\rangle\rangle = t_c^{d/2 - r/2} \frac{|N_\theta|^2}{\sqrt{\det \mathcal{G}_{ij}}} \frac{1}{\eta(q_o)^d} \sum_{\hat{n} \in \mathbb{Z}^r} e^{-\pi t_o \hat{n}^i (\mathcal{G}^{-1})^{ij} \hat{n}^j} \tag{2.22}$$

and we thus conclude that $r = \mathrm{rank}(\Lambda) = d$. We interpret this to mean that D-branes wrapped on foliating subtori of T^d cannot form Cardy states. Moreover, the minimal normalization is $N_\theta = (\det \mathcal{G}_{ij})^{1/4}$ and therefore, the overlap with the unit $u(1)^d$ character is $N_\theta = (\det \mathcal{G}_{ij})^{1/4}$. Equivalently, picking the tension out of the leading piece as $q_c \to 0$ in the closed string channel,

$$\dim \mathcal{H}_{\theta\theta} = (\det \mathcal{G}_{ij})^{1/2} . \tag{2.23}$$

For example, choosing the diagonal torus with Narain lattice $\Gamma(E) = \{\frac{1}{\sqrt{2}}(\frac{n_i}{R_i} - m_i R_i; \frac{n_i}{R_i} + m_i R_i)\}$ and rotation $\mathcal{R} = \mathrm{diag}\{-1^p; +1^{d-p}\}$ corresponding to a wrapped p-brane then one easily finds:

$$\dim \mathcal{H}_{\theta\theta} = \prod_{i=1}^{p} R_i \prod_{i=p+1}^{d} \frac{1}{R_i} \tag{2.24}$$

If one adds a flat B-field and considers a d-brane wrapping T^d then $\mathcal{R} = -E^{-1}E^{tr}$. In this case one finds

$$\dim \mathcal{H}_{\theta\theta} = (\det \mathcal{G}_{ij})^{1/2} = |\det(G_{\mu\nu} + B_{\mu\nu})| (\det G_{\mu\nu})^{-1/2} . \tag{2.25}$$

Both of these agree with standard formulae for the mass of a wrapped brane when one recalls that $1/G_{26-d} = \sqrt{\det G_{\mu\nu}}/G_{26}$. In particular, (2.25) gives the sensible result that the tension is given by the Born-Infeld action $\sqrt{|\det(G_{\mu\nu} + B_{\mu\nu})|}$, a result which is essentially already to be found in [1].

3. The general argument: superstrings

3.1 Cardy condition for $\mathcal{N} = 1$ superconformal field theory

To define the superconformal algebra we need to pick a spin structure. The index set $\mathcal{I}$ of characters divides into NS and R sectors which we denote as: $\mathcal{I}_+ := \mathcal{I}_{NS}$, $\mathcal{I}_- := \mathcal{I}_R$. Similarly, we must consider both characters and characters twisted by $(-1)^F$:

$$\chi_i^\epsilon(q) := \mathrm{Tr}_{\mathcal{H}_i}(\epsilon^F q^H) \tag{3.1}$$

where $\epsilon = \pm 1$. The transformation $\tau \to -1/\tau$ acts in a standard way on the spin structures, so we can define several modular matrices:

$$\begin{aligned}
\chi_i^+(q_c) &= \sum_{j \in \mathcal{I}_+} S_{ij}^{++} \chi_j^+(q_o) , & i \in \mathcal{I}_+ \\
\chi_i^-(q_c) &= \sum_{j \in \mathcal{I}_-} S_{ij}^{+-} \chi_j^+(q_o) , & i \in \mathcal{I}_+ \\
\chi_i^+(q_c) &= \sum_{j \in \mathcal{I}_+} S_{ij}^{-+} \chi_j^-(q_o) , & i \in \mathcal{I}_- \\
\chi_i^-(q_c) &= \sum_{j \in \mathcal{I}_-} S_{ij}^{--} \chi_j^-(q_o) , & i \in \mathcal{I}_- .
\end{aligned} \tag{3.2}$$

Note that $(S^{++})^2 = C$ is the conjugation matrix (and for simplicity we will now assume reps are self-conjugate). $S^{--}_{ij} = \delta_{ij}$ because the Witten index is modular invariant.[2]

We can form character states $|i;\eta\rangle\rangle$ in the standard way. η denotes the choice of isomorphism between left and right supercurrents so the character basis gives a basis of solutions to the linear equations

$$\left(G_r - i\eta\tilde{G}_{-r}\right)|\alpha\rangle\rangle = 0\,, \tag{3.3}$$

where $r \in \mathbb{Z}$ in the R sector and $r \in \mathbb{Z} + 1/2$ in the NS sector.

We can form Cardy states in the usual way:

$$|\alpha,\epsilon;\eta\rangle\rangle = \sum_{i\in\mathcal{I}_\epsilon} \frac{\psi_\alpha^{\ i}(\eta)}{\sqrt{S^{\epsilon,+}_{i1}}}|i;\eta\rangle\rangle\,. \tag{3.4}$$

The Cardy condition becomes

$$\langle\langle\alpha,\epsilon;\eta|q_c^{\frac{1}{2}(L_0+\tilde{L}_0-\frac{c}{12})}|\beta,\epsilon;\eta'\rangle\rangle = \mathrm{Tr}_{\mathcal{H}^{\eta\eta'}_{\alpha\beta}}\,\epsilon^F q_o^{H_o} = \sum_{j\in\mathcal{I}_{\eta\eta'}} (n_j^{\epsilon,\eta\eta'})_\beta^{\ \alpha}\chi^\epsilon_j(q_o)\,. \tag{3.5}$$

In a standard unitary theory we will require that $(n_j^{\epsilon,\eta\eta'})_\alpha^{\ \beta} \geq 0$.

In particular, focusing on the $(-,-)$ spin structure, which is invariant under $\tau \to -1/\tau$, we have:

$$(n_i^{++})_\alpha^{\ \beta} = \sum_{j\in\mathcal{I}_+} \frac{S^{++}_{ij}}{S^{++}_{1j}}\psi^j_\alpha(\eta)(\psi^j_\beta(\eta))^*\,. \tag{3.6}$$

We find again that

$$\dim\mathcal{H}^{NS}_{\alpha\alpha} = \dim\mathcal{H}^{\eta,\eta}_{\alpha,\alpha} = \left(\frac{\psi_\alpha^{\ 1}(\eta)}{\sqrt{S^{++}_{11}}}\right)^2 = \sum_{j\in\mathcal{I}_+}(n_j^{++})_\alpha^{\ \alpha}S^{++}_{j1}\,. \tag{3.7}$$

Remarks:

1. Equation (3.3) is only one of several choices one must make in the isomorphism of left and right chiral algebras.

2. In (3.4) we are considering a Cardy set with states with fixed η and which are either purely NS or purely R. To make GSO invariant states it is necessary to combine states with different η as described below. We can also form Cardy sets with states which are linear combinations of character states from the R and NS sector, and this is required in order to obtain BPS boundary states. There are only minor modifications to the analysis.

[2]Warning: We are using a possibly confusing piece of notation. χ_i^+ for $i \in \mathcal{I}_+$ is the path integral for the spin structure commonly denoted $(-,-)$!

3. In (3.7) ${\psi_\alpha}^1(\eta)$ is independent of η.

4. In (3.7) $n_j^{++} \geq 0$, and if the unit operator is present then n_j^{++} is positive definite for $j = 0$. Thus, as for the bosonic string, we can conclude that the right hand side is positive.

3.2 Tension and dimension for the superstring

We begin with a product of superconformal theories $\mathcal{C}_1(\mathbb{R}^{8-d}) \otimes \mathcal{C}_2$ where $\mathcal{C}_1(\mathbb{R}^{8-d})$ is the lightcone superconformal field theory of $8-d$ free bosons and fermions X^μ, ψ^μ. $\mathcal{C}_2$ is a unitary superconformal field theory with $\hat{c} = \hat{\tilde{c}} = d$. The closed string is a subtheory because (a) we restrict to the subspace with the same spin structures on both factors and (b) we GSO project. The statespace has the form:

$$(\mathcal{H}^1_{NS} \otimes \mathcal{H}^2_{NS})^+ \oplus (\mathcal{H}^1_R \otimes \mathcal{H}^2_R)^+ \,. \tag{3.8}$$

where the + superscript indicates the need to take a GSO projection. We will take the brane to be a point particle in the uncompactified dimensions. The general form of the boundary state is then [10]

$$\begin{aligned} C^{\alpha,NS,+}|x,NS;\eta\rangle\rangle^{(1)} \otimes |\alpha,NS;\eta\rangle\rangle^{(2)} + C^{\alpha,NS,-}|x,NS;-\eta\rangle\rangle^{(1)} \otimes |\alpha,NS;-\eta\rangle\rangle^{(2)}+ \\ +C^{\alpha,R,+}|x,R;\eta\rangle\rangle^{(1)} \otimes |\alpha,R;\eta\rangle\rangle^{(2)} + C^{\alpha,R,-}|x,R;-\eta\rangle\rangle^{(1)} \otimes |\alpha,R;-\eta\rangle\rangle^{(2)} \,. \end{aligned} \tag{3.9}$$

Here we will assume the state is made from a Cardy set of boundary states of the internal theory, although this is probably not necessary (i.e. only the combined theory really needs to obey sewing axioms). The constants are determined by imposing the GSO projection, by choosing BPS or anti-BPS branes,and by Cardy's condition. In supersymmetric type II theory the GSO projection requires $C^{\alpha,NS,-} = -C^{\alpha,NS,+}$ and $C^{\alpha,R,+} = C^{\alpha,R,-}$. Note that the GSO projection does *not* project out the unit operator in the internal theory. Other choices of GSO projection will lead to other conditions on the coefficients. For example, in type 0 theory where one has a diagonal GSO projection, $C^{\alpha,NS,+}$ and $C^{\alpha,NS,-}$ are independent, reflecting the doubling of the number of D-brane states [26].

The graviton vertex operator is $\epsilon_{\mu\nu}\psi^\mu\tilde{\psi}^\nu e^{ikX} \otimes 1$ so the coupling of the state (3.9) to the graviton is, up to a phase,

$$(C^{\alpha,NS,+} - C^{\alpha,NS,-})\frac{{\psi_\alpha}^1(\eta)}{\sqrt{S_{11}^{++}}} \,. \tag{3.10}$$

We can obtain a quantization condition on the coefficient C^{NS} in order to ensure that the combined state continues to satisfy the Cardy condition. The condition depends on whether or not we want to enforce the GSO projection in the open string channel (i.e. whether or not we consider a BPS or a non-BPS type brane). If we do not enforce the condition then $C^R = 0$ and $2|C^{NS}|^2 \in \mathbb{Z}_+$. Taking the minimal value allowed we get the open string NS sector is

$$\mathrm{Tr}_{\mathcal{H}^{NS}_{\alpha\alpha}}\, q_o^{H_o} = \frac{1}{\eta^{(8-d)/2}} \left(\frac{\vartheta_3}{\eta}\right)^{(8-d)/2} \sum_{i\in\mathcal{I}_+} n^i_{\alpha\alpha} \chi_i^+(q_o)\,. \tag{3.11}$$

If we want to have the usual GSO projection then we take a bilinear form for the cylinder amplitude[3] and impose $2(C^{NS}_{12})^2 = -(C^R_{12})^2/8 = n/2$ and taking the minimal value $n = 1$ we have

$$\mathrm{Tr}_{\mathcal{H}^{NS}_{\alpha\alpha}}\, q_o^{H_o} = \frac{1}{\eta^{(8-d)/2}} \frac{1}{2} \left[\left(\frac{\vartheta_3}{\eta}\right)^{(8-d)/2} \sum_{i\in\mathcal{I}_+} n^i_{\alpha\alpha} \chi_i^+(q_o) + \left(\frac{\vartheta_4}{\eta}\right)^{(8-d)/2} \sum_{i\in\mathcal{I}_+} n^i_{\alpha\alpha} \chi_i^-(q_o) \right]. \tag{3.12}$$

In either case, taking into account (3.10) and fixing the overall normalization from the torus case we get the superanalog of (2.3):

$$(\ell_s M)^2 = \frac{1}{16\pi^2} \frac{(2\pi\ell_s)^{8-d}}{G_{10-d}} \dim \mathcal{H}^{NS}_{\alpha\alpha}\,. \tag{3.13}$$

Here $\mathcal{H}^{NS}_{\alpha\alpha}$ is the open string channel NS sector of the full string theory. In particular, this result holds for BPS and nonBPS, the difference of a factor of $\sqrt{2}$ in the tension between a BPS and non-BPS brane [27] is due to the factor of 1/2 in the projection operator which reduces the dimension of $\mathcal{H}^{NS}_{\alpha\alpha}$ by a factor of two.

4. Discussion

We conclude with two speculative remarks on possible future applications of this work.

First, it is natural to speculate that the mass formula derived here in terms of the Affleck-Ludwig degeneracy g

$$\mathcal{M}^2 \sim |g|^2 = \mathrm{Tr}_{\mathcal{H}_{\alpha\alpha}}(1) \tag{4.1}$$

could play a role in extensions of the attractor mechanism [28] to nonsupersymmetric, spherically symmetric charged black holes. In the context of BPS black holes in $\mathcal{N} = 2$ supersymmetric compactifications to four dimensions, it is clear that g can be identified with the central charge $|\mathcal{Z}|$. The attractor mechanism tells us that we should associate spherically symmetric extremal RR charged black holes with gradient flow on the moduli space using the Zamolodchikov metric and the potential $\log|\mathcal{Z}| \sim \log(\mathrm{g})$ [29, 30]. Since g is intrinsically well defined in string theory without any reference to supersymmetry, one can conjecture that in general compactifications, one can associate spherically symmetric extremal RR charged black hole solutions

[3]Thus changing the orientation so that we have two ingoing circles. This is necesary to get the spacetime fermion minus sign.

with gradient flows using the Zamolodchikov metric and the potential log(g). The "doubly extremal" solutions (with constant values of the closed string moduli as a function of the radius) then arise when one chooses the moduli to sit at a (local) minimum of g. It would also be interesting to see if the dynamical evolution of couplings on a test 3-brane falling into a black hole is related to renormalization group flow of the boundary entropy.

Second, our result fits in well with the currently emerging interconnections between D-branes, K-theory, and noncommutative geometry. In the framework of noncommutative geometry formulae have been derived for the energy of wrapped D-branes in [31] and subsequent papers. See [32] for a recent discussion. In order to account for the mass $\mathcal{M}$ of a wrapped D-brane in its groundstate one must add to the noncommutative Yang-Mills action a term proportional to $\mathrm{Tr}_E(1)$, where E is the projective module of sections of the (noncommutative) Chan-Paton bundle. On the other hand, in the present paper we have derived a relation of the form $\mathcal{M}^2 \sim \mathrm{Tr}_{\mathcal{H}_{\alpha\alpha}}(1)$. The two results are compatible if we can identify (at least in the $\alpha' \to 0$ limit corresponding to the NCSYM theory) $\mathcal{H}_{\alpha\alpha} = \mathrm{End}(E)$, and such an identification in this limit is strongly suggested by the behavior of the finite dimensional Chan-Paton factors. Clearly, this connection deserves closer scrutiny.

It is also worth noting that in the theory of operator algebras one can define the Murray-von Neumann dimension,[4] which assigns a continuous dimension to certain Hilbert spaces of operators. It is well-known that, at least in some RCFTs, the regularized dimensions (2.17) are indeed given by ratios of such Murray-von Neumann dimensions. In this way the Jones index of subfactors appears in RCFT. On the other hand, these same dimensions can be related to traces of projection operators on towers of finite-dimensional multimatrix algebras [33]. Analogous constructions applied to the "algebra" of open string vertex operators might lead to interesting new ways of formulating D-branes, and, perhaps, even M-theory.

Acknowledgments

We would like to thank T. Banks, J. Cardy, M. Douglas, P. Fendley, D. Friedan, J. Maldacena, A. Recknagel, H. Saleur, V. Schomerus, N. Seiberg, I. Singer and E. Witten for useful discussions and remarks. We would like to acknowledge the hospitality of the Aspen Center for Physics and the Amsterdam Summer Workshop on String Theory. G.M. would like to thank the Institute for Advanced Study for hospitality and the Monell foundation for support. The work of J.H. is supported by NSF Grant No. PHY 9901194, S.K. is supported by an A.P. Sloan Foundation Fellowship and a DOE OJI Award, G.M. is supported by DOE grant DE-FG02-92ER40704, and ES is supported by an A.P. Sloan Foundation Fellowship, a DOE OJI Award, and by the DOE under contract DE-AC03-76SF00515.

[4]In the literature on operator algebras this is also called the "coupling constant".

JHEP03(2000)001

References

[1] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, *Adding holes and crosscaps to the superstring*, *Nucl. Phys.* **B 293** (1987) 83; *Loop corrections to superstring equations of motion*, *Nucl. Phys.* **B 308** (1988) 221.

[2] J. Polchinski and Y. Cai, *Consistency of open superstring theories*, *Nucl. Phys.* **B 296** (1988) 91.

[3] J.L. Cardy, *Boundary conditions, fusion rules and the verlinde formula*, *Nucl. Phys.* **B 324** (1989) 581; *Boundary conditions in conformal field theory*, talk at Taniguchi Symposium on Integrable models, October 1988; *Boundary conditions, fusion rules and the verlinde formula*, *Nucl. Phys.* **B 324** (1989) 581.

[4] D.C. Lewellen, *Sewing constraints for conformal field theories on surfaces with boundaries*, *Nucl. Phys.* **B 372** (1992) 654;
J.L. Cardy and D.C. Lewellen, *Bulk and boundary operators in conformal field theory*, *Phys. Lett.* **B 259** (1991) 274.

[5] D. Friedan, *The space of conformal field theories for the* $c = 1$ *gaussian model*, unpublished, 1994.

[6] G. Pradisi, A. Sagnotti and Y.S. Stanev, *Planar duality in* SU(2) *WZW models*, *Phys. Lett.* **B 354** (1995) 279 [hep-th/9503207];
G. Pradisi, A. Sagnotti and Y.S. Stanev, *The open descendants of nondiagonal* SU(2) *WZW models*, *Phys. Lett.* **B 356** (1995) 230 [hep-th/9506014]; *Completeness conditions for boundary operators in 2d conformal field theory*, *Phys. Lett.* **B 381** (1996) 97 [hep-th/9603097].

[7] M. Li, *Boundary states of D-branes and Dy-strings*, *Nucl. Phys.* **B 460** (1996) 351 [hep-th/9510161].

[8] P.D. Vecchia et al., *Classical p-branes from boundary state*, *Nucl. Phys.* **B 507** (1997) 259 [hep-th/9707068].

[9] M.B. Green and M. Gutperle, *Light-cone supersymmetry and D-branes*, *Nucl. Phys.* **B 476** (1996) 484 [hep-th/9604091].

[10] O. Bergman and M.R. Gaberdiel, *A non-supersymmetric open-string theory and s-duality*, *Nucl. Phys.* **B 499** (1997) 183 [hep-th/9701137].

[11] J. Fuchs and C. Schweigert, *A classifying algebra for boundary conditions*, *Phys. Lett.* **B 414** (1997) 251 [hep-th/9708141]; *Branes: from free fields to general backgrounds*, *Nucl. Phys.* **B 530** (1998) 99 [hep-th/9712257]; *Symmetry breaking boundaries, 1. General theory*, *Nucl. Phys.* **B 558** (1999) 419 [hep-th/9902132]; *Symmetry breaking boundaries, 2. More structures, examples*, hep-th/9908025.

[12] A. Recknagel and V. Schomerus, *D-branes in Gepner models*, *Nucl. Phys.* **B 531** (1998) 185 [hep-th/9712186]; *Boundary deformation theory and moduli spaces of D-branes*, *Nucl. Phys.* **B 545** (1999) 233 [hep-th/9811237]; *Moduli spaces of D-branes in CFT-backgrounds*, *Fortsch. Phys.* **48** (2000) 195 [hep-th/9903139].

[13] I. Brunner, M.R. Douglas, A. Lawrence and C. Romelsberger, *D-branes on the quintic*, hep-th/9906200.

[14] R.E. Behrend, P.A. Pearce, V.B. Petkova and J.-B. Zuber, *Boundary conditions in rational conformal field theories*, hep-th/9908036.

[15] A. Cohen, G. Moore, P. Nelson, and J. Polchinski, *An invariant string propagator*, in *Unified string theories*, M. Green and D. Gross eds., World Scientific, 1986, p. 568,.

[16] N. Ishibashi, *The boundary and crosscap states in conformal field theories*, *Mod. Phys. Lett.* **A 4** (1989) 251;
T. Onogi and N. Ishibashi, *Conformal field theories on surfaces with boundaries and crosscaps*, *Mod. Phys. Lett.* **A 4** (1989) 161.

[17] I. Affleck and A.W. W. Ludwig, *Universal noninteger 'ground state degeneracy' in critical quantum systems*, *Phys. Rev. Lett.* **67** (1991) 161.

[18] C.G. Callan Jr. and I.R. Klebanov, *D-brane boundary state dynamics*, *Nucl. Phys.* **B 465** (1996) 473 [hep-th/9511173].

[19] S. Elitzur, E. Rabinovici and G. Sarkisian, *On least action D-branes*, *Nucl. Phys.* **B 541** (1999) 246 [hep-th/9807161].

[20] J. Polchinski, *String theory*, Cambridge University Press, 1998.

[21] R. Dijkgraaf and E. Verlinde, *Modular invariance and the fusion algebra*, Presented at Annecy Conf. on Conformal Field Theory, Annecy, France, Mar 14-16, 1988. Published in Annecy Field Theory 1988:0087.

[22] G. Moore and N. Seiberg, *Naturality in conformal field theory*, *Nucl. Phys.* **B 313** (1989) 16.

[23] G. Moore and N. Seiberg, *Classical and quantum conformal field theory*, *Comm. Math. Phys.* **123** (1989) 177;
Lectures on rational conformal field theory, in *Strings '89*, Proceedings of the Trieste Spring School on Superstrings, 3-14 April 1989, M. Green et al. eds., World Scientific, 1990.

[24] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724 [hep-th/9510017].

[25] I. Affleck and A.W.W. Ludwig, *Phys. Rev.* **D 48** (1993) 7297.

[26] M. Billò, B. Craps and F. Roose, *On D-branes in type-0 string theory*, *Phys. Lett.* **B 457** (1999) 61 [hep-th/9902196].

[27] A. Sen, *Non-BPS states and branes in string theory*, hep-th/9904207.

[28] S. Ferrara, R. Kallosh and A. Strominger, $N = 2$ *extremal black holes*, *Phys. Rev.* **D 52** (1995) 5412 [hep-th/9508072].

[29] S. Ferrara, G.W. Gibbons and R. Kallosh, *Black holes and critical points in moduli space*, *Nucl. Phys.* **B 500** (1997) 75 [hep-th/9702103].

[30] G. Moore, *Arithmetic and attractors*, hep-th/9807087.

[31] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[32] B. Pioline and A. Schwarz, *Morita equivalence and T-duality (or B versus theta)*, *J. High Energy Phys.* **08** (1999) 021 [hep-th/9908019].

[33] F.M. Goodman, P. de la Harpe, and V.F.R. Jones, *Coxeter graphs and towers of algebras*, MSRI publications 14, Springer-Verlag 1989.